# 1D Galerkin Formulation

Finite Elements: Dr Colin Cotter

# Review

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- Weak form of equation

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- Method of weighted residuals
- Galerkin finite element formulation

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- Informal formulation considering the one-dimensional Poisson equation.
- Worked example using linear finite elements
- A mathematical statement of the formulation
- Finally might consider some important properties of the Galerkin formulation

# Descriptive Formulation

We consider the one-dimensional Poisson equation

$$L(u) \equiv \frac{\partial^2 u}{\partial x^2} + f = 0.$$

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If the boundary conditions stated above are applied to the Poisson we have a two-point boundary value problem and is said to be in the strong (classical) form

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Equivalent to setting the weighted residual to zero.

Integrating by parts gives

$$\int_0^1 \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dx = \int_0^1 v f \; dx + \left[ v \frac{\partial u}{\partial x} \right]_0^1 .$$

# Natural boundary conditions

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- This is a common approach in finite elements, it reduces the order of the second derivative and makes the matrix system symmetric.
- As the test functions are defined to be zero on Dirichlet boundaries we know that $v(0) = 0$.

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Neumann boundary conditions are naturally included in the formulation

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The weight or test function is also replaced by a finite expansion, and we get

$$\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u^\delta}{\partial x} dx = \int_0^1 v^\delta f \; dx + v^\delta(1) g_N$$

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Functions used in *u* are referred to as the trial functions whereas functions used in *v* are referred to as the test functions

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$$u^{\mathcal{H}}(\partial\Omega_{\mathcal{D}}) = 0, \quad u^{\mathcal{D}}(\partial\Omega_{\mathcal{D}}) = g_D.$$

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- Substitution gives

$$\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{H}} dx = \int_0^1 v^\delta f \; dx + v^\delta(1) g_N - \int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{D}} dx.$$

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where *f* is a known function and boundary conditions are

$$u(0) = g_D = 1, \quad \frac{\partial u}{\partial x}(1) = g_N = 1.$$

# Weak formulation

We start by considering the weak form

$$\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{H}} dx = \int_0^1 v^\delta f \; dx + v^\delta(1) g_N - \int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{D}} dx.$$

Link to matrix equations

# Finite Element basis

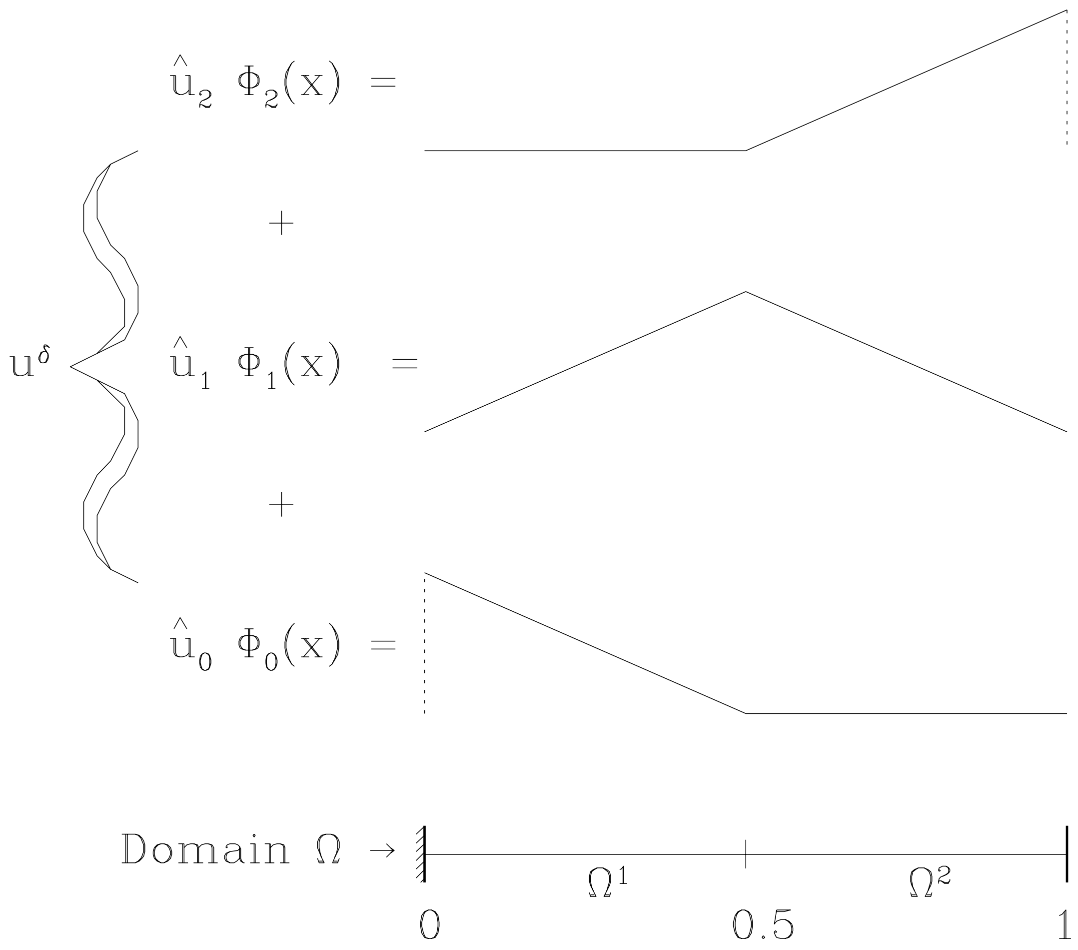
$$u^\delta = \sum_{i=0}^{2} \hat{u}_i N_i(x),$$

$$N_0(x) = \begin{cases} 1-2x & 0 \le x \le \frac{1}{2} \\ 0 & \frac{1}{2} \le x \le 1 \end{cases}$$

$$N_1(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2} \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

$$N_2(x) = \begin{cases} 0 & 0 \le x \le \frac{1}{2} \\ 2x-1 & \frac{1}{2} \le x \le 1 \end{cases}$$

# Shape functions



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- However the great power of the finite element method is its geometric flexibility arising from decomposing the global expansions into local expansions.

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To do this, we decompose $u^\delta$ into $u^\delta = u^{\mathcal{H}} + u^{\mathcal{D}}$

$$\begin{aligned} u^{\mathcal{H}} &= \hat{u}_1 N_1(x) + \hat{u}_2 N_2(x) \\ u^{\mathcal{D}} &= g_D N_0(x), \end{aligned}$$

# Galerkin discretisation

In the Galerkin approach the same expansion bases are used to define the test functions

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$$v^{\delta}(x) = \hat{v}_1 N_1(x) + \hat{v}_2 N_2(x).$$

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$$f^{\delta}(x) = \sum_{i=0}^{2} \hat{f}_i N_i(x) = \hat{f}_0 N_0(x) + \hat{f}_1 N_1(x) + \hat{f}_2 N_2(x).$$

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for our problem

$$\hat{f}_0 = f(0), \hat{f}_1 = f(0.5), \hat{f}_2 = f(1)$$

# Calculating integrals

$$
\begin{aligned}
\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{H}} dx &= \int_0^{\frac{1}{2}} (2\hat{v}_1)(2\hat{u}_1)dx + \int_{\frac{1}{2}}^1 (-2\hat{v}_1 + 2\hat{v}_2)(-2\hat{u}_1 + 2\hat{u}_2)dx \\
&= \left[\begin{array}{cc} \hat{v}_1 & \hat{v}_2 \end{array}\right] \left[\begin{array}{cc} 4 & -2 \\ -2 & 2 \end{array}\right] \left[\begin{array}{c} \hat{u}_1 \\ \hat{u}_2 \end{array}\right] \\
\int_0^1 v^\delta f \; dx &= \int_0^{\frac{1}{2}} (\hat{v}_1 2x)(\hat{f}_0(1-2x) + \hat{f}_1(2x))dx \\
&+ \int_{\frac{1}{2}}^1 (\hat{v}_1 2(1-x) + \hat{v}_2(2x-1))(\hat{f}_1 2(1-x) + \hat{f}_2(2x-1))dx \\
&= \left[\begin{array}{cc} \hat{v}_1 & \hat{v}_2 \end{array}\right] \left[\begin{array}{c} \frac{1}{12}\hat{f}_0 + \frac{1}{3}\hat{f}_1 + \frac{1}{12}\hat{f}_2 \\ \frac{1}{12}\hat{f}_1 + \frac{1}{6}\hat{f}_2 \end{array}\right] \\
v^\delta(1) g_N &= (\hat{v}_1 N_1(1) + \hat{v}_2 N_2(1)) g_N = \left[\begin{array}{cc} \hat{v}_1 & \hat{v}_2 \end{array}\right] \left[\begin{array}{c} 0 \\ 1 \end{array}\right] g_N \\
\int_0^1 \frac{\partial v^\delta}{\partial x}\frac{\partial u}{\partial x}^{\mathcal{D}} dx &= \int_0^{\frac{1}{2}} (2\hat{v}_1)(-2g_D)dx = \left[\begin{array}{cc} \hat{v}_1 & \hat{v}_2 \end{array}\right] \left[\begin{array}{c} -2g_D \\ 0 \end{array}\right].
\end{aligned}
$$

link to weak form

# Matrix equations

$$\begin{bmatrix} \hat{v}_1 & \hat{v}_2 \end{bmatrix} \left\{ \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} - \begin{bmatrix} \frac{1}{12}\hat{f}_0 + \frac{1}{3}\hat{f}_1 + \frac{1}{12}\hat{f}_2 \\ \\ \frac{1}{12}\hat{f}_1 + \frac{1}{6}\hat{f}_2 \end{bmatrix} \right.$$

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$$\left. - \begin{bmatrix} 0 \\ g_N \end{bmatrix} + \begin{bmatrix} -2g_D \\ 0 \end{bmatrix} \right\} = 0.$$

This equation has to be true for all test functions, so we get the matrix equation in the curly brackets

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$$\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} 2 + \frac{1}{12}\hat{f}_0 + \frac{1}{3}\hat{f}_1 + \frac{1}{12}\hat{f}_2 \\ \\ 1 + \frac{1}{12}\hat{f}_1 + \frac{1}{6}\hat{f}_2 \end{bmatrix}$$

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which has a solution

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} + \frac{1}{24}\hat{f}_0 + \frac{5}{24}\hat{f}_1 + \frac{1}{8}\hat{f}_2 \\ \\ 2 + \frac{1}{24}\hat{f}_0 + \frac{1}{4}\hat{f}_1 + \frac{5}{24}\hat{f}_2 \end{bmatrix}.$$

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is

$$u^{\delta} = \begin{cases} 1 + x + \dfrac{x}{12}\hat{f}_0 + \dfrac{5x}{12}\hat{f}_1 + \dfrac{x}{4}\hat{f}_2 & 0 \leq x \leq \frac{1}{2} \\ 1 + x + \dfrac{1}{24}\hat{f}_0 + \dfrac{2+x}{12}\hat{f}_1 + \dfrac{1+4x}{24}\hat{f}_2 & \frac{1}{2} \leq x \leq 1 \end{cases}$$

# Mathematical Formulation

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To be solved in the domain $0 < x < l$

The equation comes with boundary conditions

$$u(0) = g_D, \quad \frac{\partial u}{\partial x}(l) = g_N.$$

# Integral form

Multiplying by an arbitrary test function, the properties of which are to be defined, and integrating over the domain we obtain

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If the functions are smooth enough, we can integrate by parts

$$\int_0^l \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \int_0^l \lambda vu \; dx = \int_0^l vf \; dx + \left[v\frac{\partial u}{\partial x}\right]_0^l.$$

# Bilinear form notation

$$
\begin{aligned}
a(v,u) &= \int_0^l \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \lambda v u\right) dx \\
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We write the equation

$$a(v,u) = f(v)$$

# Strain energy

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Functions that belong to the energy space are called $H^1$ functions and satisfy the condition that the integral of the square of the function and its derivative are bounded.

# Trial and test functions

For our problem the trial space is defined by

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The space of all test functions which are homogeneous on Dirichlet boundaries is

$$\mathcal{V} = \{v \mid v \in H^1, v(0) = 0\}.$$

# Weak formulation

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Find $u \in \mathcal{X}$ such that

$$a(v, u) = f(v), \qquad \forall\ v \in \mathcal{V}.$$

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We select subspaces $\mathcal{X}^\delta \subset \mathcal{X}, \quad \mathcal{V}^\delta \subset \mathcal{V}$
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The approximate form of the weak formulation is

Find $u^\delta \in \mathcal{X}^\delta$ such that

$$a(v^\delta, u^\delta) = f(v^\delta) \qquad \forall\, v^\delta \in \mathcal{V}^\delta.$$

# Galerkin formulation

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Find

$$u^\delta = u^{\mathcal{D}} + u^{\mathcal{H}}, \text{ where } u^{\mathcal{H}} \in \mathcal{V}^\delta,$$

such that

$$a(v^\delta, u^{\mathcal{H}}) = f(v^\delta) - a(v^\delta, u^{\mathcal{D}}) \text{ for all } v^\delta \in \mathcal{V}^\delta$$

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a(v, u) &= a(u, v) \\
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$$|a(v,u)| \leq C_1 ||v||_1 ||u||_1, \quad C_1 < \infty$$

The equation is *elliptic* if

$$a(u,u) \geq C_2 ||v||_1^2, \quad C_2 > 0$$

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Subtracting gives

$$a(v^\delta, u_1) - a(v^\delta, u_2) = a(v^\delta, u_1 - u_2) = 0$$

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Subtracting gives

$$a(v^\delta, u_1) - a(v^\delta, u_2) = a(v^\delta, u_1 - u_2) = 0$$

Now choose $v^\delta = u_1 - u_2$

$$0 = a(u_1 - u_2, u_1 - u_2)\|u_1 - u_2\|_E$$

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$$a(v^\delta, u_2) = f(v^\delta), \qquad \text{for all } v^\delta \in \mathcal{V}^\delta.$$

Subtracting gives

$$a(v^\delta, u_1) - a(v^\delta, u_2) = a(v^\delta, u_1 - u_2) = 0$$

Now choose $v^\delta = u_1 - u_2$

$$0 = a(u_1 - u_2, u_1 - u_2) \| u_1 - u_2 \|_E$$

**Contradiction!**

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Gives the result.

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So the error is minimised over the trial space in the energy norm

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$$u_1^\delta(x) = u_2^\delta(x) \quad \Rightarrow \quad \sum_{i=0}^{P} \alpha_i \psi_i(x) = \sum_{i=0}^{P} \beta_i h_i(x).$$

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- Different choices of polynomial expansion bases can have an important effect on the numerical conditioning of matrix systems.